# N-Dimensional Rocket Science

**March 2023 - Langara Physics & Astronomy**

## 0 Introduction

Shania Twain once said, "so, you're a Rocket Scientist - that don't impress me much" - well Shania, who can blame you! If regular old 3-dimensional Rocket Science can't impress Canada's second-best-selling-artist-of-all-time, we shall have to switch things up. What would the physics of rocket science look like in other dimensions? To what extent does the number of spatial dimensions affect our physical laws? As we shall see, the expression for the force of gravity is a direct result of the fact we live in three spatial dimensions, and must be amended if we are to perform calculations in two, or twenty, dimensions. Prepare to blast off to $dim = \infty$, and beyond!

## 1 Finding $\mathbf{F_{G,2}}$

*Difficulty:* 2/5

Using our understanding of Newtonian gravity's dependence on spatial dimension we learned about in the Background Theory section on page 3:

> ***Question***: Write down the formula for the force of gravity between one mass $\mathbf{M}$ and another $\mathbf{m}$ whose centres of mass are separated by a distance $\mathbf{r}$ in a flat **2-dimensional space**.

## 2 It's All g

*Difficulty:* 1.5/5

Imagine a 2-dimensional version of the Earth. Instead of being a 3-sphere (i.e. a "regular" sphere), we will make the easy generalization and say it that in 2-dimensions it takes the shape of a 2-sphere (i.e. a **circle**). We will say that this *2-Earth* has the same mass as its 3-dimensional counterpart, $\mathbf{M_\oplus}$, as well as being defined by the same radius, $\mathbf{R_E}$. One useful property of our 3-Earth is it's $\boldsymbol{g}$ - the gravitational acceleration at its surface. 3-Earth's 'g' is roughly $9.81\frac{m}{s^2}$.

> ***Question***: Find the value of the gravitational acceleration at the surface of our 2-Earth, $\mathbf{g_2}$.

## 3 Finding $\mathbf{U_{G,n}}$

*Difficulty:* 4.5/5 + *Calculus*

> ***Question***: Find the formula for the force of gravity between one mass $\mathbf{M}$ and another $\mathbf{m}$ whose centres of mass are separated by a distance $\mathbf{r}$ in a flat **n-dimensional space**; integrate this function to find the **gravitational potential energy**, $\mathbf{U_{G,n}}$ for $\mathbf{n \geq 1}$, that the mass $\mathbf{m}$ has when situated $\mathbf{r}$ away from $\mathbf{M}$. Note that this calculation is fully analogous to the 3-dimensional derivation of $\mathbf{U_{G,3}}$. You should expect a correct answer to be defined piecewise.

## 4 Deriving the n-Dimensional Escape Velocity

*Difficulty:* 3/5

Imagine an n-dimensional Earth, for $n \geq 1$. As in **Question 2**, imagine this *n-Earth* to be an n-sphere of mass $\mathbf{M}_{\oplus}$ and radius $\mathbf{R_E}$. Imagine that the n-CSA (n-dimensional Canadian Space Agency) is attempting to launch a rocket into deep space, and want to know what velocity their rocket will need to achieve before it will be able to escape from the n-Earth's pull.

***Question***: Calculate the n-dimensional escape velocity $\mathbf{v_{e,n}}$ of a rocket of mass $\mathbf{m}$ from an *n-Earth*. Is $\mathbf{v_{e,n}}$ ever infinite, for some n, making it *impossible* to escape that n-Earth's pull? Is $\mathbf{v_{e,n}}$ ever 0?

## 5 Submitting Your Solutions, & Prizes

Your completed solutions must be either: put in Hand-In Slot #10 outside room T340; handed to me (Alex); or emailed to achoinski@langara.ca before midnight on March 31st. Depending on how many problems you correctly solve, you will get either one,two,three, or four entries into a draw for a **prize to be awarded at the end of the month!**. In addition, you will be placed on our **Display Case Leaderboard**, whose first place entry will win a **grand prize** at the end of the semester! Questions about the contest or problems can be directed to achoinski@langara.ca.

# 6 Background Theory

## 6.1 Geometry of Newton's Gravity

Here is a non-rigorous explanation of the origin of the inverse-square law for gravity. The result at the end, however, **is** rigorously true.

The gravitational field follows an inverse-square relation because, in some sense, it is conserved: there's only so much of it emanating from the source, and no extra erupts from empty space as you travel away from the source. You're stuck with no more than the strength of field you start with. If we imagine a source mass **M**, with a gravitational field radiating outward from the source point in all directions, the surface of points of equal field strength will form a sphere. We know that the surface area of a sphere is given by $A = 4\pi r^2$, so as the field gets farther and farther away from its source, it gets *spread out* over an increasingly large total area that grows proportional to $r^2$. From this we might deduce that the intensity of the gravitational field **must *fall*** according to $\mathbf{\frac{1}{r^2}}$, in order for the total amount of field strength at some distance **r** to remain constant for all **r**. In the figure on the next page, we see that the total gravitational field passing through a surface **per unit area** falls as an inverse square law, precisely due to the fact that surface area grows by $r^2$.

## 6.2 Higher Dimensions & n-Spheres

Mathematicians decided long ago that restricting ourselves to three dimensions was far too boring. For our purposes, we will see here that it is rather straightforward to generalize the concept of a sphere to other dimensions. Let's take the definition of a sphere to be: "The set of all points some distance r away from the origin"- what would this look like in 2-d, close to our ordinary 3-d home? If you think about it, the set of all points some distance $r$ away from the origin on the 2-d plane defines a circle! "Circle" is actually just a nice term for a *2-sphere* - i.e. the 2-dimensional *version* of a sphere. (Actually, technically, it's a 1-sphere, but don't worry about that!). A *normal* sphere as we know it is just a *3-sphere*. But there's no need to stop there...

We can say the following: For an n-sphere (sphere in n dimensions), it's *volume* will go as $r^n$. Check if this is true with the familiar examples of $n = 2$ and $n = 3$ - does the area of a circle grow like $r^2$? Does the volume of a sphere grow like $r^3$?. Now think about surface area: what is the relationship between the power on the radius $r$ and number of dimensions $n$ for surface area in 2 and 3 dimensions...? The answer generalizes! The surface area of an n-sphere is indeed proportional to $r^{n-1}$. There are good, deep reasons for that, and if you're curious you can ask me all about it.

We have seen that the force of gravity experienced at some distance $r$ away is proportional to $\frac{1}{r^2}$ due to the fact that the *surface area* of a sphere in 3d is a function of $r^2$, **but this is only in three dimensions**. This relationship between radius and surface area will change as we travel to new dimensions, and thus, so will Newtonian gravity's relation with distance.

As a starter, here is the force of gravity between a mass **m** and another mass **M**, in *four* flat spatial dimensions. See if you can work out how I got there.S

$$|\mathbf{F_{G,4}}| = \frac{GmM}{r^3} \tag{1}$$

**This is all the physics (and *more*) that you will need to solve this month's problems.**

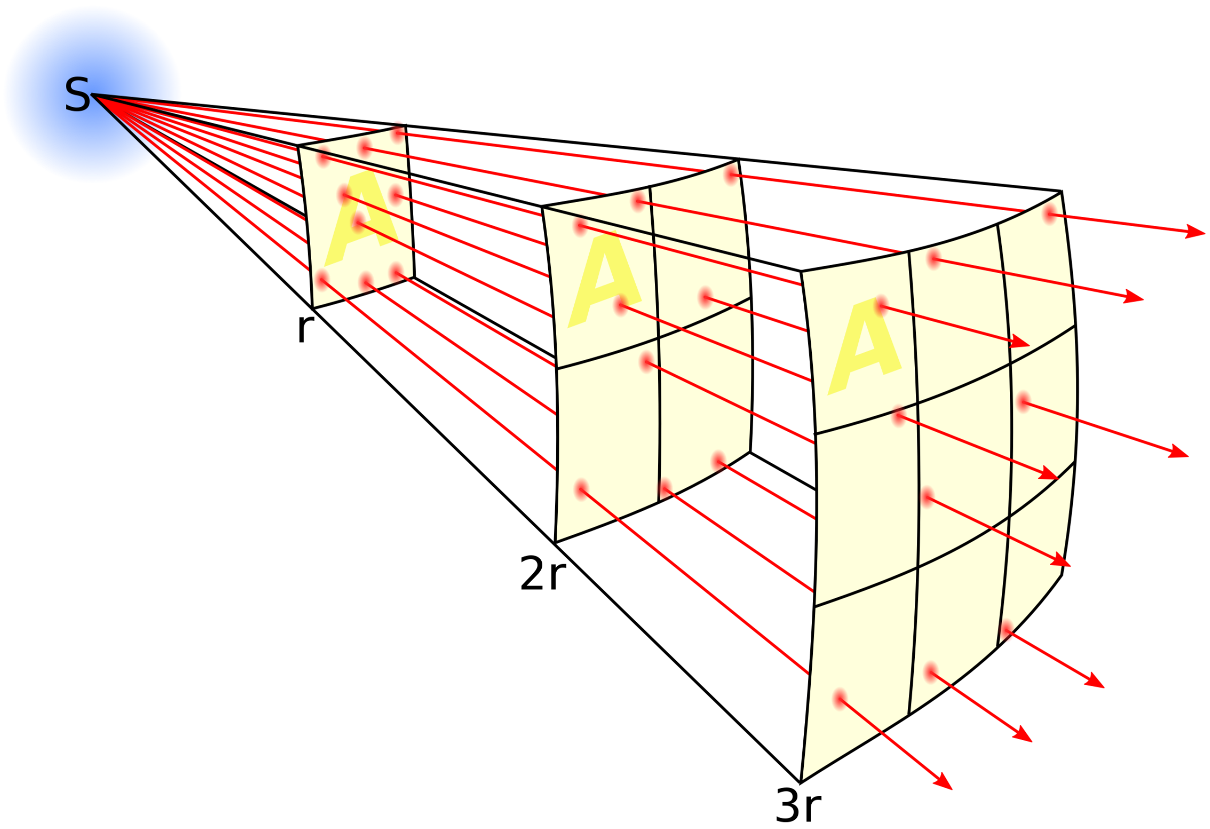


Figure 1: This image shows gravitational field lines emanating outward from a source S. Three cross sections of this field are made, showing the same number of field lines passing through successively larger areas, implying that the field strength per unit area decreases as an inverse to the square of the distance. Retrieved from:
`https://commons.wikimedia.org/wiki/File:Inverse_square_law.svg`