Uncertainty Propagation Examples

Example 1: Find δq, where $q = A^2BC$. The values and the uncertainties of A, B and C are known positive numbers.

Solution:

$$q = A^2BC = AABC$$

Using the "Multiplication and Division" rule, the relative uncertainty in the product is the sum of the relative uncertainties of the constituents:

$$\frac{\delta q}{q} = \frac{\delta A}{A} + \frac{\delta A}{A} + \frac{\delta B}{B} + \frac{\delta C}{C} = 2\frac{\delta A}{A} + \frac{\delta B}{B} + \frac{\delta C}{C}$$

To find the absolute uncertainty from the relative uncertainty:

$$\delta q = q\left(\frac{\delta q}{q}\right)$$

Substitute in the numbers to find the numerical value of δq.

Example 2: Find δm, where $m = m_1 + 2m_2$. The values and the uncertainties of m_1 and m_2 are known.

Solution:

Using the "Addition or Subtraction" rule, we add the absolute uncertainties

$$\delta m = \delta m_1 + \delta(2m_2)$$

Using the "Multiplication and Division" rule

$$\frac{\delta(2m_2)}{2m_2} = \frac{\delta 2}{2} + \frac{\delta m_2}{m_2} = \frac{\delta m_2}{m_2} \quad \text{(because } \delta 2 = 0\text{)},$$

so $\delta(2m_2) = \frac{\delta m_2}{m_2}(2m_2) = 2\delta m_2$,

and finally $\delta m = \delta m_1 + \delta(2m_2) = \delta m_1 + 2\delta m_2$

Note that we can also get $\delta(2m_2) = 2\delta m_2$ directly by using the "Multiplication by a Constant" rule.

Example 3: The mass of a cylinder is measured to be $m = (112.3 \pm 0.1)$ g. The diameter is measured to be $d = (1.23 \pm 0.01)$ cm and the length is measured to be $l = (5.77 \pm 0.01)$ cm. What is the density ρ of the cylinder and its uncertainty?

Calculations

Convert unit:

$m = (112.3 \pm 0.1) \times 10^{-3}$ kg, $d = (1.23 \pm 0.01) \times 10^{-2}$ m, $l = (5.77 \pm 0.01) \times 10^{-2}$ m.

So the density is:

$$\rho = \frac{m}{V} = \frac{m}{\pi r^2 l} = \frac{4m}{\pi d^2 l} = \frac{4\left(112.3\times10^{-3}\,\text{kg}\right)}{\pi\left(1.23\times10^{-2}\,\text{m}\right)^2\left(5.77\times10^{-2}\,\text{m}\right)} = 16380\,\text{kg/m}^3$$

Uncertainty Analysis

$$\begin{aligned}\frac{\delta\rho}{\rho} &= \frac{\delta m}{m} + 2\frac{\delta d}{d} + \frac{\delta l}{l}\\ &= \frac{0.1\text{g}}{112.3\text{g}} + 2\left(\frac{0.01\text{cm}}{1.23\text{cm}}\right) + \frac{0.01\text{cm}}{5.77\text{cm}}\\ &= 0.018884\end{aligned}$$

This is from the multiplication rule. Note that $\delta\pi/\pi = 0$ and $\delta 4/4 = 0$ because they are exact numbers.

percentage uncertainty: $\frac{\delta\rho}{\rho} = (0.018884)(100\%) = 1.8884\% \rightarrow 2\%$

absolute uncertainty: $\delta\rho = \rho\left(\frac{\delta\rho}{\rho}\right) = (16380\ \text{kg/m}^3)(0.018884) = 309.32\ \text{kg/m}^3$

Conclusions

The density of the cylinder, ρ, was found to be $(1.64 \pm 0.03) \times 10^4$ kg/m^3 (±2%).

(Note that because the uncertainty $\delta\rho$ is bigger than 10 and we want to keep only 1 sig fig for it, we have to use scientific notation to report the final result.)